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Distributed order equations as boundary value problems

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ABSTRACT

There has been a recent increase in interest in the use of distributed order differential equations (particularly in the case where the derivatives are given in the Caputo sense) to model various phenomena. Recent papers have provided insights into the numerical approximation of the solution, and some results on existence and uniqueness have been proved. In each case, the representation of the solution depends, among other parameters, on Caputo-type initial conditions. In this paper we discuss the existence and uniqueness of solutions and we propose a numerical method for their approximation in the case where the initial conditions are not known and, instead, some Caputo-type conditions are given away from the origin.

1. Introduction

In this paper, we shall consider distributed order linear equations of the form

$$\int_0^m \beta(r) D^r y(t) dr = f(t), \quad 0 \le t \le T \tag{1}$$

where $D^r y(t)$ is the derivative of $y(t)$ in the Caputo sense, that is

$$D^r y(t) = {}^{RL}D^r (y - T[y])(t),$$

$T[y]$ is the Taylor polynomial of degree $\lfloor r \rfloor$ for y centred at 0, and ${}^{RL}D^r$ is the Riemann–Liouville derivative of order r, given by

$${}^{RL}D^r := D^{\lceil r \rceil} J^{\lceil r \rceil - r}$$

with $J^\gamma y$ being the Riemann–Liouville integral operator defined by:

$$J^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) ds.$$

We shall also assume, without loss of generality, that $m > 0$ is an integer; if m is not an integer, we can always define $\beta(r) = 0$ on the interval $(m, \lceil m \rceil]$ and use $\lceil m \rceil$ instead of m in (1). Where appropriate, we extend the functions f and β to functions defined on $\mathbb{R}^+$ by setting their values to zero outside the intervals $[0, T]$ and $[0, m]$, respectively.

We assume further the following:

(H1) β is an absolutely integrable function on the interval $[0, m]$ and satisfies

$$\int_0^m \beta(r) s^r dr \neq 0, \quad \text{for } Re(s) > 0,$$

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(H2) $f \in L^1[0, \infty)$,
(H3) y is such that $D^r y < M$ for $t \in [0, \infty)$ and for every $r \in [0, m]$.

In recent decades, equations of the form (1) have been used to model several phenomena, for example, the stress–strain behavior of an elastic medium [1], to find the eigenfunctions of the torsional models of anelastic or dielectric spherical shells and infinite plates [2], to model dielectric induction and diffusion [3], to model the input–output relationship of a linear time-variant system based on frequency domain observations [4,5], to study rheological properties of composite materials [6,7] and to model ultraslow and lateral diffusion processes [8].

In [9], Caputo provided the basic analysis for simple distributed order equations of the form

$$\int_\alpha^\gamma \beta(r) D^{m+r} y(t) dr = f(t),$$

where $0 < \alpha < \gamma < 1$ and $m \in \mathbb{N}$. This distributed order equation is, in fact, a particular case of Eq. (1), where there are no integer order derivatives, since in this case all the orders of the derivatives belong to the interval $(m, m+1)$. In [10] Diethelm and Ford, proceeding similarly to Caputo in [9] established the existence and uniqueness of the solution of Eq. (1) and proved the following theorem:

Theorem 1. *If conditions* (H1)–(H3) *hold, then Eq.* (1) *has a unique solution given by:*

$$y(t) = y(0) + \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^m \beta(r)(\cdot)^r dr} \right] \right)(t) + \sum_{k=1}^{m-1} y^k(0) \mathcal{L}^{-1}\left[\frac{\int_k^m \beta(r)(\cdot)^{r-k-1} dr}{\int_0^m \beta(r)(\cdot)^r dr} \right](t), \tag{2}$$

where $*$ denotes the standard convolution operation

$$(g * h)(t) = \int_0^t g(t-\tau) h(\tau) d\tau,$$

for suitable functions g and h, and $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Expression (2) gives an indication of the ways in which the solution depends on the right-hand side function f, on the Caputo-type initial conditions $y^k(0)$, $k = 0, \dots, m-1$ and on β, the distribution of the derivatives.

In [11,10] the authors also proposed a numerical method to solve the distributed order differential equation (1), given the initial conditions $y^k(0)$, $k = 0, \dots, m-1$.

Here we will follow a similar approach but we assume that the Caputo-type initial conditions are not known and instead, the following boundary conditions are given

$$y^{(k)}(a) = y_a^k, \quad k = 0, \dots, m-1, \text{ for a certain } a, 0 < a \le T. \tag{3}$$

If $a = T$ then (3) defines a boundary condition in the usual sense. If $a < T$ then one can regard (3) as a *generalised boundary condition* or *interior condition*.

The paper is organised in the following way: because distributed order differential equations may be considered as generalisations of single or multi-term fractional differential equations, in Section 2 we briefly review the recently obtained results for single and multi-term fractional boundary value problems. In Section 3 we discuss the existence and uniqueness of the solution of problem (1) and (3). In Section 4 we propose a numerical method to approximate the solution of the distributed order equation. We finish by presenting some numerical results that illustrate the good performance of our proposed method and we end with some conclusions.

2. Single and multi-term fractional differential equations

As pointed out in [11,10], distributed order differential equations may be regarded as a generalisation of single-term fractional differential equations

$$D^r y(t) = f(t, y(t))$$

or multi-term fractional differential equations

$$\sum_{i=0}^{k} \gamma_i D_i^r y(t) = f(t, y(t)), \quad 0 < r_1 < r_2 < \dots < r_k.$$

Therefore, in this section, we will briefly recall the main results obtained for single and multi-term fractional differential equations as boundary value problems.

Concerning single-term fractional boundary value problems (FBVPs) of the form

$$D_*^\alpha y(t) = f(t, y(t)), \qquad y(a) = y_a, \quad a > 0,$$

we have proved recently [12] that if

(i) $0 < \alpha < 1$,
(ii) the function f is continuous and satisfies a Lipschitz condition with Lipschitz constant $L > 0$ with respect to its second argument,
(iii) $\frac{2La^\alpha}{\Gamma(\alpha+1)} < 1$,

then the FBVP is equivalent to the following integral equation

$$y(t) = y(a) - \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s, y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds.$$

We have also proved that if conditions (i)–(iii) are satisfied, then the solution of the FBVP exists and is unique on a certain interval $[0, T]$, $T \geq a$.

In that paper we also used a shooting algorithm as in Ref. [13] to find the value of $y(0)$ for which the solution of the initial value problem

$$D_*^\alpha y(t) = f(t, y(t)), \qquad y(0) = y_0,$$

satisfies the boundary condition $y(a) = y_a$.

Very recently, in [14], we extended these results to multi-term FBVPs of the form

$$D^\alpha y(t) = f(t, y(t), D^{\beta_1} y(t), D^{\beta_2} y(t), \ldots, D^{\beta_n} y(t)), \tag{4}$$

$$y^{(k)}(a) = y_a^{(k)}, \quad k = 0, 1, \ldots, \lceil \alpha \rceil, a > 0, \tag{5}$$

where $\alpha > \beta_1 > \beta_2 > \cdots > \beta_n \in \mathbb{Q}$.

Defining M as the least common multiple of the denominators of $\alpha, \beta_1, \ldots, \beta_n$, $\gamma = 1/M$ and $N = \alpha M$ we proved that the multi-term FBVP (4)–(5) is equivalent to the following system of N equations

$$\begin{aligned} D^\gamma y_1(t) &= y_2(t) \\ D^\gamma y_2(t) &= y_3(t) \\ &\vdots \\ D^\gamma y_{N-1}(t) &= y_N(t) \\ D^\gamma y_N(t) &= f\left(t, y_{\frac{\beta_1}{\gamma}+1}(t), \ldots, y_{\frac{\beta_n}{\gamma}+1}(t)\right), \end{aligned} \tag{6}$$

together with conditions

$$y_j(a) = \begin{cases} y_a^{(k)}, & \text{if } j = kM + 1 \text{ for some } k \in \mathbb{N} \\ y_a^{(j)}, & \text{else}, \end{cases} \tag{7}$$

in the following sense: whenever $Y = (y_1, \ldots, y_N)^T$ with $y \in C^{\lceil \alpha \rceil}[0, T]$, for some $T \geq a$, is the solution of the system of Eqs. (6), then the function $y = y_1$ solves the multi-term Eq. (4) and satisfies the boundary conditions (5); on the other hand, whenever $y \in C^{\lceil \alpha \rceil}[0, T]$ is a solution of the multi-term Eq. (4) satisfying the boundary conditions (5), then $Y = (y_1, \ldots, y_N)^T = \left(y, D^\gamma y, D^{2\gamma} y, \ldots, D^{(N-1)\gamma} y,\right)^T$ satisfies the system (6) and the conditions (7), for suitable constants $y_a^{(j)}$. It should be noted that the values of $y_a^{(j)}$ are known only in the cases where $j = kM + 1$ for some $k \in \mathbb{N}$, and that the remaining ones are not known.

Taking this into account, and because each of the equations in this system is a single-term fractional differential equation with order $0 < \gamma < 1$, we proved easily that if

(iv) $\alpha, \beta_1, \ldots, \beta_n \in \mathbb{Q}$,
(v) the function f satisfies a uniform Lipschitz condition, with Lipschitz constant L, in all its arguments except for the first on a suitable domain D,
(vi) $\frac{2La^\gamma}{\Gamma(\gamma+1)} < 1$,

then, the multi-term FBVP (4)–(5) has a unique continuous solution on an interval $[0, T]$ of the real line, $T \geq a$.

With respect to the numerical approximation of the solution of (4)–(5), we also proposed a shooting algorithm based on the equivalence between the FBVP (4)–(5) and the corresponding system of equations. As explained fully in that paper, we considered the initial value problem

$$\begin{aligned} D^\gamma y_1(t) &= y_2(t) \\ D^\gamma y_2(t) &= y_3(t) \\ &\vdots \\ D^\gamma y_{N-1}(t) &= y_N(t) \\ D^\gamma y_N(t) &= f\left(t, y_{\frac{\beta_1}{\gamma}+1}(t), \ldots, y_{\frac{\beta_n}{\gamma}+1}(t)\right), \end{aligned}$$

together with conditions

$$y_j(0) = \begin{cases} y_0^{(k)}, & \text{if } j = kM + 1 \text{ for some } k \in \mathbb{N} \\ 0, & \text{else,} \end{cases}$$

where the values of $y_j(0)$ are not known, whenever $j = kM + 1$, for some $k \in \mathbb{N}$. For the remaining cases we will have $y_j(0) = 0$ due to the following lemma proved in [15]:

Lemma 2. *Let $y \in C^k[0, T]$ for some $T > 0$ and some $k \in \mathbb{N}$, and let $0 < q < k$, $q \notin \mathbb{N}$. Then $D^q y(0) = 0$.*

We have then fixed the unknowns $y_j(0)$ and $y_j(a)$ in order to ensure that the solution of the initial value problem satisfies the boundary conditions imposed at $t = a$.

3. Existence and uniqueness of solution

An important aspect of this paper is the need to establish whether the boundary value problem has a unique solution. It is tempting to concentrate on constructing a numerical scheme without having regard to the fundamental need for an existence and uniqueness theory which, among other things, establishes that the problem is well-posed.

First, we consider the case where $m = 1$ and we assume that hypotheses (H1)–(H3) hold. That is, we consider the following problem

$$\int_0^1 \beta(r) D^r y(t) dr = f(t), \quad 0 \le t \le T \tag{8}$$

$$y(a) = y_a \tag{9}$$

and we will show that there will exist a unique initial value $y(0)$ for which the solution of (8) satisfies condition (9).

Let $\beta_n(r)$ be a sequence of piecewise constant functions defined in $[0, 1]$, such that $\lim \beta_n(r) = \beta(r)$ a.e. on $[0, m]$ and let us consider the sequence of distributed order problems

$$\int_0^1 \beta_n(r) D^r y_n(t) dr = f(t), \quad 0 \le t \le T \tag{10}$$

$$y_n(a) = y_a. \tag{11}$$

For each n, $n = 0, 1, \ldots,$ we can use a quadrature scheme to approximate (10) using certain quadrature weights ω_j and nodes $z_j \in [0, 1]$, by

$$\sum_{j=0}^{l} \omega_j \beta_n(z_j) D^{z_j} y_n(t) = f(t), \tag{12}$$

which leads to a multi-term fractional differential equation. Note that, as pointed out in [10], Eq. (10) can be viewed as the limiting case of Eq. (12), where a very large number of terms is considered. As explained in the previous section, this multi-term Eq. (12) can be written as a single-term system of equations and then we can find the unique initial value, say y_{n0}, such that the solution of the corresponding initial value problem satisfies (11). Obviously the multi-term fractional boundary value problem must satisfy the needed conditions to ensure the existence of the initial value, as explained in the previous section.

On the other hand, by (2), for each n, $n = 0, 1, \ldots,$ the solution of (10)–(11) is given by

$$y_n(t) = y_n(0) + \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^1 \beta_n(r)(\cdot)^r dr} \right] \right)(t). \tag{13}$$

Next we show that the limit of sequence of the initial values $y_n(0) = y_{n0}$ exists.

For $t = a$ and for two distinct values of n, $n = n_1$ and $n = n_2$, we obtain from (13)

$$\left| y_{n_1}(0) - y_{n_2}(0) \right| \le \left| y_{n_1}(a) - y_{n_2}(a) \right| + \left| \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^1 \beta_{n_1}(r)(\cdot)^r dr} \right] \right)(a) - \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^1 \beta_{n_2}(r)(\cdot)^r dr} \right] \right)(a) \right|.$$

Letting $n_1, n_2 \to \infty$, the right-hand side of this inequality vanishes and therefore we obtain

$$\left| y_{n_1}(0) - y_{n_2}(0) \right| \to 0,$$

that is, the limit of the sequence of the initial values exists since $y_{n_1}(0)$ and $y_{n_2}(0)$ must tend to the same value, say $y_0 = y(0)$, and this is the initial value to use for problem (8)–(9).

In order to prove the uniqueness for the initial value for the distributed order equation, let us assume that there exist two initial value problems, y_0 and $\tilde{y}_0$, for the distributed order Eq. (8) whose corresponding solutions, y and $\tilde{y}$, satisfy the condition (9). We will have, according to (2):

$$y(a) - \tilde{y}(a) = y(0) - \tilde{y}(0),$$

that is, in order to have $y(a) = \tilde{y}(a)$ we must have $y(0) = \tilde{y}(0)$ proving the uniqueness of the initial value.

Now we consider the case $m > 1$. It is clear that we can construct a sequence of functions β_n in a similar way to the above. For each β_n there will be corresponding initial values $y_n(0), y_n'(0), \ldots$ etc.. We need to show that this set of initial values tends to a unique limiting set of initial values. This is intuitively clear since the mapping from the function β_n to the corresponding initial values appears to be continuous. However we have not seen a formal proof of this property and for the moment we should therefore regard the existence and uniqueness theory for $m > 1$ as a conjecture.

4. Numerical method and results

In this section we present a numerical method to approximate the solution of problems of the form (1) and (3). The idea is to follow the approach used in the previous papers [11,10]. First, we discretise the integral term in the distributed order equation using a quadrature formula. As we will see, this will result on a multi-term fractional differential equation satisfying the boundary conditions imposed at $t = a$, (3). Then, we apply a numerical method to solve the resulting multi-term FBVP as explained in Section 2.

As was pointed out in [10], when we have integer orders of the derivatives we will need to take into account the jumps of $D^r y(t)$, when r is an integer, if we intend to use, for example, the trapezium rule to approximate the integral term in the distributed order equation. In that paper [10] the authors proved the following result

Lemma 3. *Let $y \in C^p[0, T]$ with some $p \in \mathbb{N}$ and $T > 0$. For every fixed $t \in (0, T]$, consider $D^r y(t) = z(r)$ as a function of r. Then*

- *z is a C^∞ function on $\cup_{j=1}^{p}(j, j+1]$;*
- *At the integer argument $j = 1, 2, \ldots, p-1$ the function z has a jump discontinuity, since*

$$\lim_{r->j^+} z(r) - \lim_{r->j^-} z(r) = -y^{(j)}(0).$$

Note that there is no jump discontinuity if and only if $y^{(j)}(0) = 0$.

4.1. Example 1

We consider the following example:

$$\begin{aligned} &\int_{0.1}^{0.9} \Gamma(3-r) D^r y(t) dr = 2\frac{t^{1.9} - t^{1.1}}{\ln t}, \quad 0 \le t \le 1 \\ &y(0.1) = 0.01 \end{aligned} \tag{14}$$

whose analytical solution is $y(t) = t^2$.

First, we approximate the integral term using, for example, the trapezium rule with $n = 4$, obtaining in this way the following multi-term fractional differential equation:

$$\begin{aligned} &0.1\Gamma(2.9)D^{0.1}y(t) + 0.2\Gamma(2.7)D^{0.3}y(t) + 0.2\Gamma(2.5)D^{0.5}y(t) \\ &\quad + 0.2\Gamma(2.3)D^{0.7}y(t) + 0.1\Gamma(2.1)D^{0.9}y(t) = 2\frac{t^{1.9} - t^{1.1}}{\ln t}. \end{aligned}$$

Second, we reduce this multi-term equation into a system of single-term equations (in this case the dimension of the system is 9). Taking into account the conditions imposed at $t = a$, we obtain the following system

$$\begin{aligned} &D^{0.1}y_1(t) = y_2(t) \\ &D^{0.1}y_2(t) = y_3(t) \\ &\vdots \\ &D^{0.1}y_8(t) = y_9(t) \\ &D^{0.1}y_9(t) = \frac{1}{\Gamma(2.1)}\left(20\frac{t^{1.9} - t^{1.1}}{\ln t} - \Gamma(2.9)y_2(t) - 2\Gamma(2.7)y_4(t) - 2\Gamma(2.5)y_6(t) - 2\Gamma(2.3)y_8(t)\right), \end{aligned} \tag{15}$$

Table 1
Approximate values of $y(0)$ and $y_{ia} = D^{0.1(i-1)}y(0.1)$, $i = 2, \ldots, 9$, for Example 1.

	$h = 1/50$	$h = 1/100$	$h = 1/200$
y_0	0.00909119	0.00487313	0.00278968
y_{2a}	0.0021405	0.00800545	0.0104757
y_{3a}	0.00676407	0.0127599	0.0152367
y_{4a}	0.0124399	0.0189632	0.0216511
y_{5a}	0.0250221	0.0291448	0.0309939
y_{6a}	0.038381	0.0413565	0.0430041
y_{7a}	0.065095	0.0618707	0.0607522
y_{8a}	0.0882269	0.0837794	0.0820193
y_{9a}	0.141256	0.123258	0.114878

together with the conditions

$$y_1(0.1) = 0.01 \tag{16}$$

$$y_i(0.1) = y_{ia}, \quad i = 2, \ldots, 9. \tag{17}$$

Finally, we solve the initial value problem formed by Eqs. (15) and conditions

$$y_1(0) = y_0$$

$$y_i(0) = 0, \quad i = 2, \ldots, 9.$$

using, for example, the Adams rule (see [16]) or the backward difference method proposed in [17]. We finally adjust the unknown values of y_{ia}, $i = 2, \ldots, 9$, and y_0 in order that the obtained solution satisfies the conditions (16)–(17).

As in the case where the initial conditions are given, the numerical approximation of problems of the form (1) and (3) has several contributions to the error: one has its origin in the choice of quadrature formula and a second one in the chosen initial value problem solver. The interaction between these errors was discussed fully in [10]. Moreover, in this case, since the solution of the initial value problem is obtained with an approximated value for the initial conditions, we have another error contribution, which can be considered as a small perturbation of the exact initial conditions (for the dependence of the solution on the initial given data see [18]). An optimal method would need to be chosen to have a similar magnitude of error in the initial conditions as the error given by the solution method, discussed in [10].

In Table 1 we present the obtained values of $y_0, y_{2a}, \ldots, y_{9a}$.

As expected, we observe in Figs. 1 and 2 that the absolute error decreases whenever the stepsizes of the quadrature formula and the initial value solver also decrease.

It should be remarked that in this example all the orders of the derivatives belong to the interval (0, 1). Therefore, according to Lemma 3, we have in this case no jump discontinuities in the function within the integral term.

4.2. Example 2

We now consider the following example:

$$\int_0^2 \Gamma(4-r)D^r y(t)dr = \frac{6t^3+6t-4}{\ln t} + \frac{6-10t}{\ln^2 t} + \frac{4t-4}{\ln^3 t}, \quad t \geq 0$$

$$y(0.1) = 4.201, \qquad y'(0.1) = 2.03 \tag{18}$$

whose analytical solution is $y(t) = t^3 + 2t + 4$.

Note that in this example, we will have integer order derivatives in the problem since the derivatives lie in the interval [0, 2]. According to Lemma 3 and once the exact solution is known, we will have jump discontinuities as described in that lemma since we do not have homogeneous initial conditions. Therefore we cannot expect the method used in the previous example to perform in the same way with this problem. The phenomenon may be observed in Table 2, where we have used the trapezium rule and the backward difference method to solve the multi-term fractional differential equation.

But if we insist, as was done in [10], that every integer value on the integral interval (in this case [0, 2]) is a grid point of the quadrature formula, we can take into account the jumps described in Lemma 3 by using the so-called modified trapezium rule explained in [10]. Some numerical results for this example, using this quadrature method, combined once again with the backward difference method to solve the obtained multi-term differential equation are reported in Table 3.

4.3. Example 3

Finally we consider the following example:

$$\int_0^2 \frac{\Gamma(6-r)}{120} D^r y(t)dr = \frac{t^5-t^3}{\ln t}, \quad t \geq 0$$

$$y(0.2) = 0.00032, \qquad y'(0.2) = 0.008 \tag{19}$$

whose analytical solution is $y(t) = t^5$.

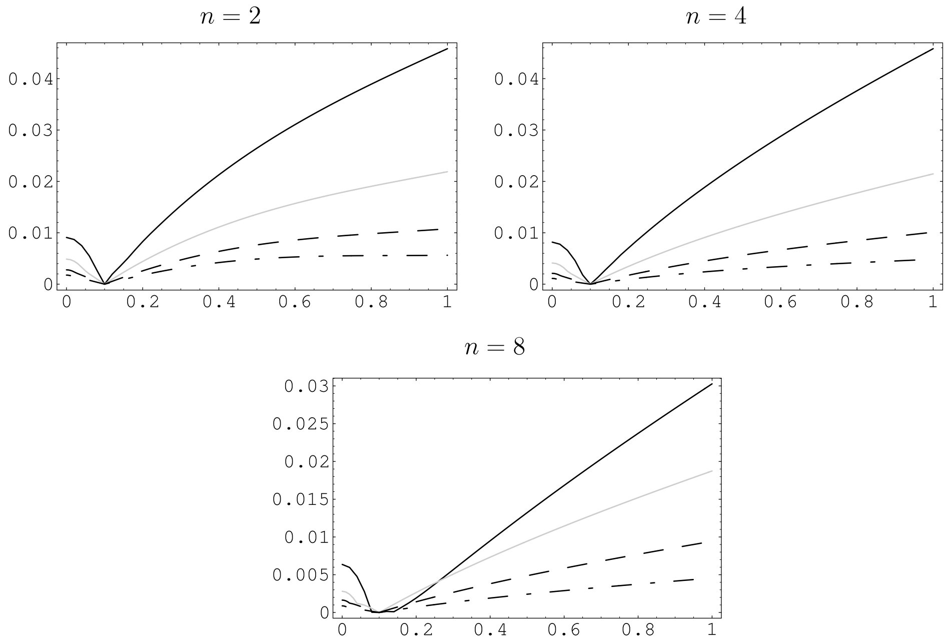

Fig. 1. Absolute errors at the discretisation points trapezium rule with n subintervals and Adams rule with step sizes 1/50 (black line), 1/100 (gray line), 1/200 (dashed line) and 1/400 (dot-dashed line), for Example 1.

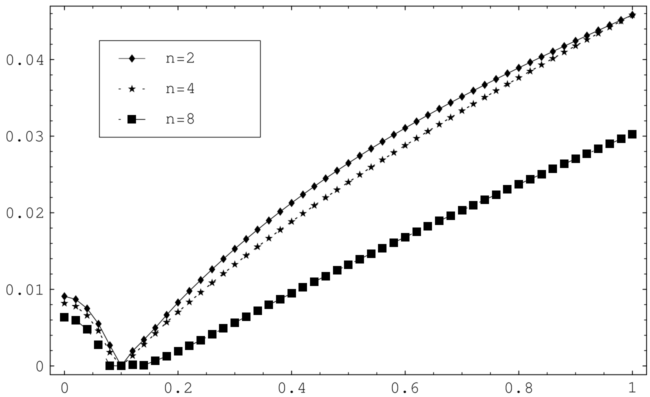

Fig. 2. Absolute errors at the discretisation points (trapezium rule with n subintervals and Adams rule with step size 1/50) for Example 1.

Table 2
Absolute errors in $y(0.5)$ for Example 2 using the standard trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	1.26385	0.535302	0.253611
0.05	1.28935	0.536039	0.245587
0.025	1.30012	0.532421	0.240642
0.0125	1.30487	0.529266	0.238199
0.00625	1.30706	0.527318	0.236982
0.003125	1.30810	0.526228	0.236357

In this case, we will also have integer order derivatives and since the analytical solution is known we conclude that we will have no jump discontinuities since the initial conditions are homogeneous. Anyway, since usually we do not know in general whether or not we will have homogeneous initial conditions, for prudence, we should use the modified trapezium rule if we have integer order derivatives, since the numerical results in this case are similar if you use the standard or the modified trapezium rule, as can be observed in Tables 4 and 5.

Table 3
Absolute errors at $y(0.5)$ for Example 2 using the modified trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	0.0512000	0.00542023	0.00015709
0.05	0.0286896	0.00603624	0.00418375
0.025	0.0174860	0.00535740	0.00329024
0.0125	0.0119594	0.00451476	0.00215495
0.00625	0.0092399	0.00397546	0.00152362
0.003125	0.0079010	0.00368779	0.00121633

Table 4
Absolute errors in $y(0.5)$ for Example 3 using the standard trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	0.0840826	0.0104716	0.0079724
0.05	0.0455065	0.0037384	0.0026886
0.025	0.0299246	0.0010759	0.0008104
0.0125	0.0230027	0.0000523	0.0001718
0.00625	0.0197510	0.0003317	0.0000381
0.003125	0.0181764	0.0004731	0.0001055

Table 5
Absolute errors in $y(0.5)$ for Example 3 using the modified trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	0.0838955	0.0103730	0.0079349
0.05	0.0454385	0.0037065	0.0026761
0.025	0.0298956	0.0010708	0.0008069
0.0125	0.0229889	0.0000577	0.0001715
0.00625	0.0197438	0.0003223	0.0000374
0.003125	0.0181723	0.0004622	0.0001044

5. Conclusions

In this paper we have investigated the existence, uniqueness and the approximation of the solution of distributed order equations in the case where the initial values for the solution are not known. Results on existence and uniqueness had been established in the previous works of Caputo and the papers of Diethelm and Ford, all of them providing a representation of the solution in terms of the initial values. Based on some previous recent papers where we have studied boundary value problems for fractional differential equations, we have established the existence and uniqueness results by showing that a similar representation of the solution can be obtained in the case where the initial values are not given. Based on quadrature we have also proposed a numerical scheme in order to approximate the solution of the considered problems.

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